

Some comments on developments in exact solutions in statistical mechanics since 1944

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Abstract. Lars Onsager and Bruria Kaufman calculated the partition function of the Ising model exactly in 1944 and 1949. Since then there have been many developments in the exact solution of similar, but usually more complicated, models. Here I shall mention a few, and show how some of the latest work seems to be returning once again to the properties observed by Onsager and Kaufman.

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1. Introduction

The first solution of a finite-dimensional lattice model that exhibits a phase transition was the calculation by Lars Onsager in 1944 of the partition function of the zero-field square-lattice Ising model.[1] In 1949 Bruria Kaufman derived this result using spinor or free-fermion operators.[2]

We discuss the history of the derivation of the spontaneous magnetization \mathcal{M} of the Ising model in section 4: it depended on first using the free fermion structure to write \mathcal{M} , for a finite lattice, in terms of a determinant, then taking the limit when the lattice and the dimensions of the determinant become infinite.[3, 4, 5]

After the Ising model, and the related dimer problems, the next models to be solved were the six-vertex and eight-vertex models. We discuss these in sections 5 and 6, in particular we outline the Bethe ansatz calculation for the general six-vertex model in a field. Then many other solvable models were found, both two and three-dimensional.[6]-[15] These were more complicated than the Ising model, lacking its simple free-fermion structure and explicit determinantal expressions for correlations such as the magnetization. (One exception is the six-vertex model with particular boundary conditions.[16])

One of the most challenging has been the two-dimensional N -state solvable chiral Potts model.[17] It lacks the “rapidity-difference” property, so there is no explicit single-valued parametrization of the Boltzmann weights of the model, and

the simple inversion relation and corner transfer matrix tricks to calculate the free energy and magnetization fail.

However, there is a special “superintegrable” case[18] of the chiral Potts model whose magnetization is that of the general model and which has simple properties similar to those of the Ising model. Indeed, when $N = 2$ it *is* the Ising model.

In particular it has recently been shown[19]-[23] that its magnetization can be expressed as an m by m determinant D , even for a finite lattice.

So the wheel has come full circle and we are back to determinantal expressions. It still remains to calculate D in the large-lattice limit, when m becomes infinite. For the superintegrable chiral Potts model, the determinant is not Toeplitz, but it is Cauchy-like, so D can be evaluated explicitly and the magnetization obtained.[22]

2. Partition functions and transfer matrices

We draw the square lattice diagonally, as in Figure 1, with M rows of sites and L sites per row. On each site i we place a “spin” σ_i , which takes some set of discrete values. For the Ising model, $\sigma_i = +1$ or -1 ; for a general N -state model, $\sigma_i = 0, 1, \dots, N - 1$. Adjacent spins i, j interact with an energy $\epsilon(\sigma_i, \sigma_j)$. Spins on the top and bottom rows are fixed to the value 0 (for the Ising model, to the value $+1$), and cylindrical boundary conditions are imposed, so that we identify the spins in the extreme left column with those in the extreme right.

Let

$$W(\sigma_i, \sigma_j) = e^{-\epsilon(\sigma_i, \sigma_j)/k_B \mathcal{T}}$$

be the Boltzmann weight of the edge $\langle i, j \rangle$ (k_B is Boltzmann’s constant and \mathcal{T} the temperature). Then the partition function is

$$Z = \sum_{\sigma} \prod_{\langle i, j \rangle} W(\sigma_i, \sigma_j) , \quad (2.1)$$

where σ is the set of all spins, and the sum is over all their permitted values.

We are particularly interested in calculating the partition function per site

$$\kappa = Z^{1/LM} , \quad (2.2)$$

the dimensionless free energy $f = -\log \kappa$, and averages such as the Ising model magnetization

$$\mathcal{M} = Z^{-1} \sum_{\sigma} \sigma_1 \prod_{\langle i, j \rangle} W(\sigma_i, \sigma_j) . \quad (2.3)$$

We expect κ and \mathcal{M} to tend to limits when $L, M \rightarrow \infty$ and the central spin σ_1 becomes deep within the lattice.

Of course, such calculations are not easy: for the Ising model on a ten by ten lattice there are $2^{100} \sim 10^{30}$ terms in the sum in (2.1). One starts by defining the row-to-row transfer matrix T . Let $s = \{\sigma_1, \sigma_2, \dots, \sigma_L\}$ be the spins in one row of

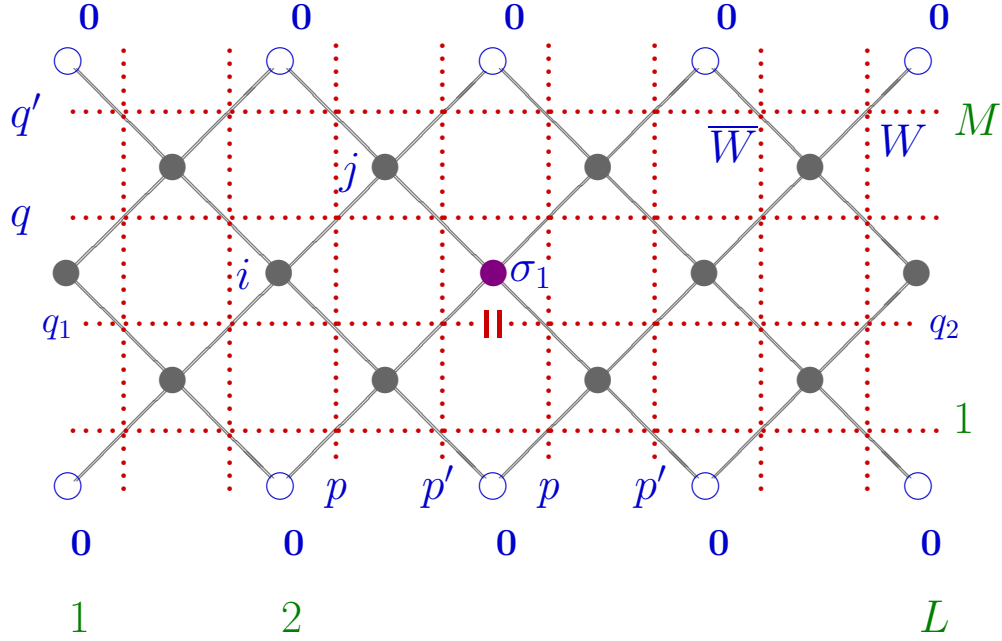


Figure 1. The square lattice turned through 45° .

the lattice, and $s' = \{\sigma'_1, \sigma'_2, \dots, \sigma'_L\}$ the spins in the row above. Define the N^L by N^L matrix T with entries

$$T_{s,s'} = \prod_i W(\sigma_i, \sigma'_i) \overline{W}(\sigma_i, \sigma'_{i-1}) , \quad (2.4)$$

taking W to be the weight on SW-NE edges and \overline{W} to be the (possibly different) weight on SE-NW edges. Then the partition function is

$$Z = u^\dagger T^M u , \quad (2.5)$$

where u is the N^L -dimensional vectors with entries

$$u_s = \delta(\sigma_1, 0) \cdots \delta(\sigma_L, 0) . \quad (2.6)$$

For M large, it follows that

$$Z \sim (\lambda_{\max})^M , \quad (2.7)$$

so we have reduced the calculation of κ to the calculation of the maximum eigenvalue of an N^L -dimensional matrix. For the above-mentioned $N = 2$, $L = 10$ case this dimension is $2^{10} = 1024$, which is a huge improvement on 10^{30} !

3. Ising model

Even so, one still wants to take the limit $L \rightarrow \infty$, so T is going to become infinite-dimensional. For the Ising model, $\sigma_i = \pm 1$ and

$$W(\sigma_i, \sigma_j) = e^{K\sigma_i\sigma_j} \quad , \quad \overline{W}(\sigma_i, \sigma_j) = e^{\overline{K}\sigma_i\sigma_j} \quad . \quad (3.1)$$

Onsager considered this model, with the more usual 90° orientation of the lattice and toroidal (cyclic) boundary conditions. Then T is the product of two matrices, one of which adds the horizontal edges within a row, while the other adds the vertical edges between rows. He showed that these two matrices generate a finite-dimensional Lie algebra, now known as the Onsager algebra.[1, eqns. 60 & 61]. This enabled him to calculate *all* the eigenvalues of T , and hence Z and κ .

The matrix T commutes with the operator R that negates all spins:

$$R_{s,s'} = \prod_{i=1}^L \delta(\sigma_i, -\sigma'_i) \quad , \quad R^2 = 1 \quad (3.2)$$

so its eigenvectors v either lie in the sub-space where $Rv = v$ or in the sub-space $Rv = -v$. In the former case they are contained in the eigenvalues of T_+ , in the latter in T_- , where

$$T_{\pm} = t_1 \otimes t_2 \otimes \cdots \otimes t_L \quad (3.3)$$

and the t_i are two-by-two matrices (the t_i are different for T_+ and T_-). So in this sense T is a direct product of L two-by-two matrices.

Bruria Kaufman later gave a simpler derivation of this result, using anti-commuting spinor (free-fermion) operators, i.e. a Clifford algebra.[2]

4. Ising model Magnetization

4.1. Kaufman and Onsager

In 1949, at a conference in Florence, Italy,[3] Onsager referred to the magnetization of the Ising model and announced that “B. Kaufman and I have recently solved” this problem. He gave the result as

$$\mathcal{M} = (1 - k^2)^{1/8} \quad (4.1)$$

where

$$k = 1/(\sinh 2K \sinh 2\overline{K}) \quad (4.2)$$

and the result is true for $0 < k < 1$. For $k > 1$ the magnetization vanishes, i.e. $\mathcal{M} = 0$, so the graph of \mathcal{M} is as in Figure 2.

Onsager and Kaufman did not publish their derivation, but we have convincing evidence of their method. In 1949 they published a paper[24] entitled “Crystal statistics. III” where they use the free-fermion spinor operators to calculate the correlation $\langle s_{1,1} s_{i,1+k} \rangle$ for $i = 1, 2$ and any integer k (*not* the k of (4.2)) in terms of

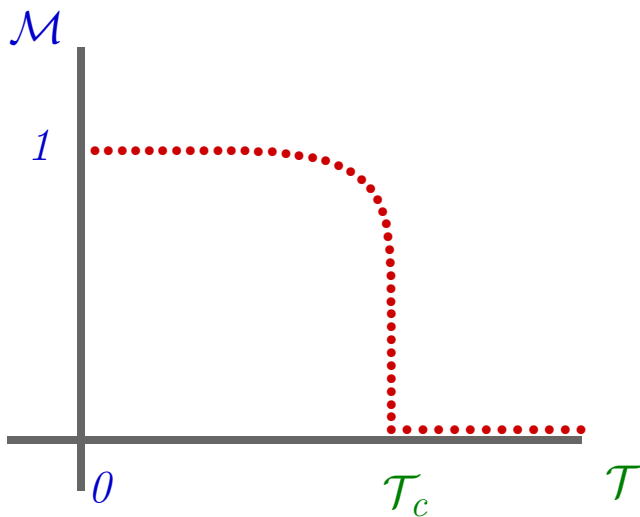


Figure 2. \mathcal{M} as a function of temperature \mathcal{T} .

a k -dimensional Toeplitz determinant Δ_k . The magnetization can be obtained from these expressions by taking the limit $k \rightarrow \infty$, when $\langle s_{1,1}s_{i,1+k} \rangle \rightarrow \mathcal{M}^2$.

So they needed to calculate the large-size limit of a Toeplitz determinant. In a letter dated April 12, 1950, Onsager wrote to Kaufman giving a general formula for this limit Δ_∞ . [25] On page 3 he says “we get the degree of order from C.S.III without much trouble. It equals $(1 - k^2)^{1/8}$ as before”. This letter is also in the Onsager archive in Trondheim, at pages 21 – 24 of

http://www.ntnu.no/ub/spesialsamlingene/tekark/tek5/research/009_0097.html

In the same archive, on pages 32, 33 of

http://www.ntnu.no/ub/spesialsamlingene/tekark/tek5/research/009_0096.html

there is a letter from Kaufman to Onsager thanking him for his letter and saying she has worked out a way of using long-range order along a row, rather than a diagonal, and applying his procedure to obtain Δ_∞ . She goes on to say that the mathematician Kakutani had written to her saying that he had spoken to Onsager about this, and was very interested.

Onsager recounts something of what happened in 1971. In [26] he says that Kaufman derived the correlations in the form of recurrent (Toeplitz) determinants, and they were particularly simple along a diagonal (i.e. a row or column of Figure 1). He then discusses the problem of calculating the infinite-size limit of these

determinants and indicates how he first did this for the particular problem by determining the eigenvalues and taking their product. This was the basis for his announcement of the result in Florence.

He then looked for a more general formula for Toeplitz determinants with arbitrary generating functions. He found one for a large class of rational generating functions.

From this, he says “the general result stared me in the face”. This is the formula (7) of [25], which is exact for his rational generating functions when the dimension k of the determinant is finite but sufficiently large. Since the elements of Δ_k are the central $2k - 1$ Laurent coefficients of the generating function, in the limit $k \rightarrow \infty$ it should be possible to extend (7) to more arbitrary functions, providing appropriate conditions for convergence are fulfilled. His last sentence reads “Only, before I knew what sort of conditions to impose on the generating function, we talked to Kakutani and Kakutani talked to Szegő and the mathematicians got there first.”

Further explanation of that comment is given in [27], where he says that he had found “a general formula for the evaluation of Toeplitz matrices.‡ The only thing I did not know was how to fill out the holes in the mathematics and show the epsilons and the deltas and all of that”. Onsager adds that six years later the mathematician Hirschman told him that he could readily have completed his proof by using a theorem of Wiener’s.

Szegő did publish his resulting general theorem[29], [30] on the large-size limit of a Toeplitz determinant, but not until 1952.

So Onsager had a derivation of \mathcal{M} in 1949, but looked for a more general way to calculate Δ_∞ . For his rational generating functions he actually proved what is now Szegő’s theorem. He could see the extension to more general functions, and that it gave the same result (4.1) as his previous method. However, he lacked a rigorous proof of that extension and did not pursue the matter when the mathematicians became interested.

The author also has a copy of a typescript, given to him by John Stephenson, entitled “Long-Range Order” and with three names hand-written at the top: “Onsager” nearest the title, “B. KAUFMAN” above that, and “R. B. Potts” above that. It shows how the magnetization can be obtained from eqn. 43 of [24], using Onsager’s working of [25] and deriving the formula of the above-mentioned letter from Kaufman to Onsager. This must be a draft of a paper by Onsager and Kaufman on the subject. The author intends to make this typescript available at <http://arxiv.org/>.

‡ Refs. [26], [27] are reprinted in Onsager’s collected works, pages 232 – 241 and 37 – 45, respectively.[28]

4.2. Yang and others

The first derivation of \mathcal{M} published was in 1952 by C.N. Yang.[4] He first wrote \mathcal{M} in terms of the two maximal eigenvectors of the transfer matrix. He then used Kaufman's spinor operators to write \mathcal{M} as the product of eigenvalues (i.e. a determinant) of an L by L matrix and evaluated the eigenvalues in the limit $L \rightarrow \infty$.

Later, combinatorial ways were found of writing the partition function of the Ising model on a finite lattice directly as a determinant or a pfaffian (the square root of an anti-symmetric determinant).[31, 32]. Then it was realised that the problem could be solved by first expressing it as one of filling a planar lattice with dimers.[33, 34, 35, 36] In 1963 Montroll, Potts and Ward[5] showed that the magnetization could be written as the ratio of two determinants, thereby obtaining a combinatorial proof of the result (4.1). The numerator determinant is indeed Toeplitz and they evaluated its large-size limit by using Szegő's theorem.

5. Six-vertex model

For any N -state model (with nearest neighbour interactions) on a lattice of L columns, the transfer matrix T is of dimension N^L . So is the hamiltonian H of a quantum mechanical system of N -state spins on a line of L sites. Calculating the largest eigenvalue of T corresponds to calculating the lowest energy state of H .

In 1966 C.N. Yang and C.P. Yang[37] extended the Bethe ansatz by using it to calculate the ground-state energy of the hamiltonian of the anisotropic Heisenberg chain, also known as the XXZ chain. The following year Elliott Lieb used this method to calculate κ for three particular models: ice, F and KDP.[6, 7, 8] These are all special cases of a more general zero-field "ice-type" or "six-vertex" model. The solution of this model was given in the same year by Sutherland.[9] Finally, the solution of the general six-vertex model was given by C. P. Yang[38] and Sutherland *et al* [39].

In these models one places arrows on the edges of the square lattice so that at each vertex there are two arrows in and two arrows out (this is known as the "ice rule"). There are six ways of doing this, as shown in the upper diagram in Figure 3. In general we assign to these six vertex configurations the weights $\omega_1, \dots, \omega_6$, as in the figure.

One can place a line on any horizontal edge bearing a left-pointing arrow and a line on any vertical edge bearing a down-pointing arrow; other edges one regards as empty. The six possible vertex configurations are then represented as in the lower diagram of Figure 3, using dots to represent empty edges. Separating the lines in the second vertex as indicated, it becomes apparent that these lines are continuous, and can be viewed as moving generally up and to the right through the lattice.

If we impose cyclic boundary conditions, then it follows that the number of lines in one row of vertical edges must be the same as the number in the row above.

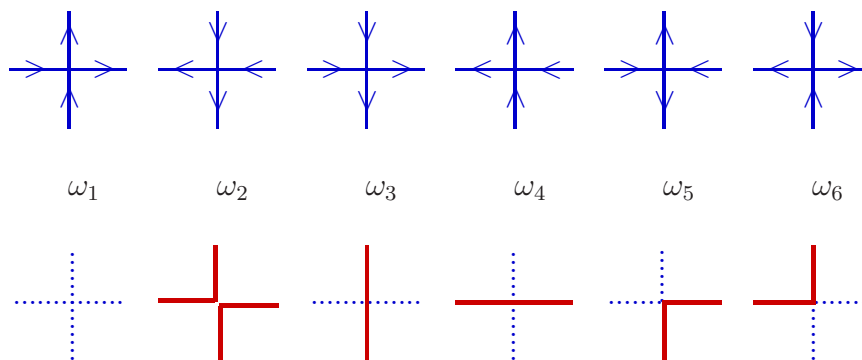


Figure 3. The six arrangements of arrows at a vertex in the six-vertex model .

(Lines can move out at the right-hand boundary, but must re-appear at the left.) Hence if there are n lines in one row, then there are n lines in all rows. This is the “line-conservation” or “conservation of down arrows” property. The row-to-row transfer matrix T therefore breaks up into $L+1$ diagonal blocks, one for each possible value $(0, 1, \dots, L)$ of n . Two typical arrangements of lines in two adjacent rows are shown in Figure 4.



Figure 4. The two typical arrangements of lines in adjacent rows of the six-vertex model. The y_1, \dots, y_n must interlace the x_1, \dots, x_n .

5.1. The Bethe ansatz

Here I shall briefly outline how the Bethe ansatz is applied to the usual row-to-row transfer matrix. In some ways it is simpler if one instead uses the transfer matrix in the diagonal SW to NE direction, since then lines only move at most one position at a time. Alternatively, helical boundary [40] conditions mean that one only has to add a single vertex at a time.

However, it is worth the effort to work with the row-to-row matrix, since the final equations simplify and can be compared with hamiltonian calculations such those mentioned above by Yang. Further, we shall find that they manifest the important commutation property of the transfer matrices.

The method is discussed in [41, 42] and, for the zero-field case (when $\omega_1 = \omega_2$ and $\omega_3 = \omega_4$), in sections 8.2 - 8.4 of [43]. Here I outline the extension to arbitrary $\omega_1, \dots, \omega_6$.

Let Λ be an eigenvalue of the transfer matrix T and f the corresponding eigenvector. Each element of f is associated with a configuration of arrows or lines on a row of vertical edges of the lattice. Write $f(x_1, \dots, x_n)$ for the element corresponding to lines (down arrows) at positions x_1, \dots, x_n be $f(x_1, \dots, x_n)$. Write the set x_1, \dots, x_n as X , and y_1, \dots, y_n as Y . We must have

$$1 \leq x_1 < x_2 < \dots < x_n \leq L \quad (5.1)$$

and similarly for Y .

Then the eigenvalue equation is

$$\Lambda f(X) = \sum C(X, Y) f(Y) + \sum D(X, Y) f(Y) \quad , \quad (5.2)$$

where the first sum is over all y_1, \dots, y_n such that

$$x_1 \leq y_1 \leq x_2 \leq y_2 \leq \dots \leq x_n \leq y_n \leq L \quad ,$$

and the second sum is over

$$1 \leq y_1 \leq x_1 \leq y_2 \leq x_2 \leq \dots \leq y_n \leq x_n \quad .$$

Here X is the configuration of arrows in one row, and Y is the configuration in the row above, as in Figure 4. The coefficients $C(X, Y), D(X, Y)$ are the products of the Boltzmann weights of the vertices in between. Note that special cases occur when an x_i is equal to a y_j , as then we can get a vertex of type 2 or type 3, with weight ω_2 or ω_3 . We must also ensure that we do not include cases where two of the y_i are equal, e.g. $y_1 = x_2 = y_2$ in the first sum.

5.2. The case $n = 1$

When $n = 1$ the equation (5.2) is (writing x_1, y_1 simply as x, y)

$$\Lambda f(x) = \omega_1^{L-1} \omega_3 f(x) + \omega_5 \omega_6 \sum_{y=x+1}^L \omega_1^{L+x-y-1} \omega_4^{y-x-1} f(y)$$

$$+ \omega_5 \omega_6 \sum_{y=1}^{x-1} \omega_1^{x-y-1} \omega_4^{L+y-x-1} f(y) + \omega_2 \omega_4^{L-1} f(x) . \quad (5.3)$$

We try the solution

$$f(x) = z^x$$

and find the RHS of (5.3) is

$$(\omega_1^L \mathcal{L} + \omega_4^L \mathcal{M}) z^x + \rho(z) \omega_1^x \omega_4^{L-x} (1 - z^L) , \quad (5.4)$$

where

$$\begin{aligned} \rho(z) &= \frac{\omega_5 \omega_6 z}{\omega_1 (\omega_1 - \omega_4 z)} \\ \mathcal{L} &= \mathcal{L}(z) = \frac{\omega_1 \omega_3 + (\omega_5 \omega_6 - \omega_3 \omega_4) z}{\omega_1 (\omega_1 - \omega_4 z)} \\ \mathcal{M} &= \mathcal{M}(z) = \frac{\omega_1 \omega_2 - \omega_5 \omega_6 - \omega_2 \omega_4 z}{\omega_4 (\omega_1 - \omega_4 z)} . \end{aligned} \quad (5.5)$$

Note that \mathcal{M} in this section is *not* the same as the magnetization \mathcal{M} discussed elsewhere in this paper. It follows that (5.3) is satisfied if

$$\Lambda = \omega_1^L \mathcal{L}(z) + \omega_4^L \mathcal{M}(z) \quad \text{and} \quad z^L = 1 .$$

Thus there are L possible values for z (all lying on the unit circle), giving the L eigenvalues Λ in the $n = 1$ block of the transfer matrix.

5.3. The case $n = 2$

When $n = 2$ we first try

$$f(x_1, x_2) = z_1^{x_1} z_2^{x_2} .$$

Then various terms come from the sums represented by the two diagrams in Figure 4. Considered as functions of x_1, x_2 , the only ones proportional to $z_1^{x_1} z_2^{x_2}$ are

$$\left[\omega_1^L \mathcal{L}(z_1) \mathcal{L}(z_2) + \omega_4^L \mathcal{M}(z_1) \mathcal{M}(z_2) \right] z_1^{x_1} z_2^{x_2} ,$$

so these terms in the eigenvalue equation are satisfied if

$$\Lambda = \omega_1^L \mathcal{L}(z_1) \mathcal{L}(z_2) + \omega_4^L \mathcal{M}(z_1) \mathcal{M}(z_2) .$$

We shall see that the contributions from the boundary terms lead to the “conservation of momentum” equation

$$z_1^L z_2^L = 1 . \quad (5.6)$$

Both these last two “energy” and “momentum” equations are unchanged by permuting z_1 and z_2 , so we could more generally try an ansatz of the form

$$f(x_1, x_2) = A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_2^{x_1} z_1^{x_2} ,$$

where A_{12} and A_{21} are coefficients that are at our disposal. Then the boundary terms contain factors $\rho(z_1)$ or $\rho(z_2)$ and vanish if

$$A_{12} = z_1^L A_{21} , \quad A_{21} = z_2^L A_{12} ,$$

from which (5.6) follows.

The other terms that arise are “unwanted internal terms” terms from the first diagram of Figure 4 containing a factor $(z_1 z_2)^{x_2}$ and from the second diagram containing a factor $(z_1 z_2)^{x_1}$. Both of these vanish if

$$s(z_1, z_2)A_{12} + s(z_2, z_1)A_{21} = 0 \quad , \quad (5.7)$$

where $s(z_1, z_2) = \omega_1 \omega_4 \mathcal{M}(z_1) \mathcal{L}(z_2) - \omega_2 \omega_3$. Removing factors that cancel out of (5.7), we can take

$$s(z_1, z_2) = \frac{\omega_1 \omega_3 - (\omega_1 \omega_2 + \omega_3 \omega_4 - \omega_5 \omega_6) z_2 + \omega_2 \omega_4 z_1 z_2}{\omega_1 \omega_3} \quad . \quad (5.8)$$

5.4. Arbitrary n

It turns out that these results generalise easily to arbitrary n , becoming

$$f(X) = \sum_P A_P z_{p_1}^{x_1} \cdots z_{p_n}^{x_n} \quad , \quad (5.9)$$

where the sum is over all $n!$ permutations $P = \{p_1, p_2, \dots, p_n\}$ of the integers $\{1, \dots, n\}$ and $A_P = A_{p_1, \dots, p_n}$. This is the Bethe ansatz. The unwanted internal terms give the equations

$$s(z_{p_j}, z_{p_{j+1}})A_P + s(z_{p_{j+1}}, z_{p_j})A_{P^{(j)}} \quad , \quad 1 \leq j < n \quad (5.10)$$

where $P^{(j)}$ differs from P in the interchange of p_j with p_{j+1} . There are many more equations than unknowns in (5.10), but they have the solution

$$A_P = \epsilon_P \prod_{1 \leq i < j \leq n} s(z_{p_j}, z_{p_i}) \quad , \quad (5.11)$$

where $\epsilon_P = +1$ for even permutations, -1 for odd permutations.

The boundary terms give

$$A_P = z_{p_1}^L A_{p_2, \dots, p_n, p_1} \quad (5.12)$$

Eliminating all A_P between these two equations, we get the n “Bethe equations”

$$z_j^L = (-1)^{n-1} \prod_{m=1, m \neq j}^n \frac{s(z_m, z_j)}{s(z_j, z_m)} \quad . \quad (5.13)$$

These imply $z_1^L \cdots z_n^L = 1$ and in general define z_1, \dots, z_L . There are many solutions, corresponding to the different eigenvalues. Finally, the wanted terms in the eigenvector equation give the eigenvalue as

$$\Lambda = \omega_1^L \mathcal{L}(z_1) \cdots \mathcal{L}(z_n) + \omega_4^L \mathcal{M}(z_1) \cdots \mathcal{M}(z_n) \quad . \quad (5.14)$$

For the ice model, where $\omega_1 = \cdots = \omega_6$, the full working is given in Ref. [6]

5.5. Free-fermion case

Note that when

$$\omega_1\omega_2 + \omega_3\omega_4 - \omega_5\omega_6 = 0$$

the function $s(z_1, z_2)$ in (5.8) is symmetric, so (5.13) simplifies to

$$z_j^L = (-1)^{n-1} ,$$

and the Bethe equations can be solved explicitly for all the eigenvalues of T .

This is known as the “free-fermion” case. Like the Ising model, it can also be solved combinatorially by pfaffians.[44]

5.6. Zero-field case

The zero-field case is when

$$\omega_1 = \omega_2 = a , \quad \omega_3 = \omega_4 = b , \quad \omega_5 = \omega_6 = c , \quad (5.15)$$

a, b, c being positive parameters. The vertices 5 and 6 always occur in pairs, being sinks and sources of horizontal arrows, so there is no restriction in taking $\omega_5 = \omega_6$. The first two conditions are definitely constraints. Then from (5.8),

$$s(z_m, z_j) = 1 - 2\Delta z_j + z_m z_j , \quad (5.16)$$

where

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab} . \quad (5.17)$$

It turns out to be useful to express these equations in terms of new variables μ, v, β_j defined by

$$a = \sin v , \quad b = \sin(\mu - v) , \quad c = \sin \mu ,$$

$$z_j = \frac{\sin \beta_j}{\sin(\mu - \beta_j)} .$$

This fixes the normalization of a, b, c and ensures that

$$\Delta = -\cos \mu , \quad \mathcal{L}(z_j) = -\frac{\sin(v - \beta_j - \mu)}{\sin(v - \beta_j)} , \quad \mathcal{M}(z_j) = -\frac{\sin(v - \beta_j + \mu)}{\sin(v - \beta_j)} .$$

From (5.16),

$$s(z_m, z_j) = G_{mj} \sin(\beta_j - \beta_m + \mu)$$

where $G_{mj} = G_{jm} = \sin \mu / [\sin(\mu - \beta_j) \sin(\mu - \beta_m)]$. The factors G_{mj} cancel out of the RHS of (5.13), so it depends on β_1, \dots, β_n only via their differences.

If we define

$$\phi(v) = \sin^L v , \quad q(v) = \prod_{j=1}^n \sin(v - \beta_j) , \quad (5.18)$$

and note that Λ is a function of v , so write it as $t(v)$, then (5.13) and (5.14) become (for $j = 1, \dots, n$)

$$\phi(\beta_j) q(\beta_j - \mu') + \phi(\mu - \beta_j) q(\beta_j + \mu') = 0 \quad (5.19)$$

$$t(v)q(v) = \phi(v) q(v - \mu') + \phi(\mu - v) q(v + \mu') \quad (5.20)$$

where $\mu' = \mu - \pi$. When $-1 < \Delta < 1$, we can choose μ, v to be positive, with $0 < \mu < \pi$, $0 < v < \mu$.

For $\Delta < 1$ and L even, the maximum eigenvalue lies in the block with

$$n = L/2$$

and the z_1, \dots, z_n lie on the unit circle, distributed about the point $z = 1$. In the thermodynamic limit of L large they form a continuous distribution and (5.13) becomes a linear integral equation for the distribution function. If we transform from the z_j to the β_j (the β_j lie on the vertical line $Re(\beta_j) = \mu/2$ in the complex plane), then this equation has a difference kernel and the integral is over a full period of the functions, so the equation can be solved by Fourier transforms, giving an exact expression for the free energy $\log \kappa$. For the F model, Lieb found[7] that there was a transition of infinite order at $\Delta = -1$, i.e. all derivatives of the free energy exist but there is an essential singularity of type $e^{-C/|\mathcal{T}-\mathcal{T}_c|^{1/2}}$, \mathcal{T} being the temperature, \mathcal{T}_c the critical temperature and C a constant. For the KDP model he found[8] there was a first-order transition at $\Delta = 1$ from a disordered phase ($\Delta < 1$), to a frozen ordered state ($\Delta > 1$) where the vertical (horizontal) arrows all point in the same direction. This was very different behaviour from that of the Ising model, which has a logarithmic singularity in the specific heat.

5.7. The model in a field

The above working remains valid when electric fields are applied, making $\omega_1 \neq \omega_2$ and/or $\omega_3 \neq \omega_4$. The system is then either frozen or the maximum eigenvalue occurs when z_1, \dots, z_n are distributed along a curve in the complex plane. The resulting integral equation cannot in general be solved analytically by Fourier transforms. However, it can be solved numerically and at (and sometimes about) various special cases (including the free-fermion case).[41], [43, §8.12], [45, 46, 47, 48]

5.8. Matrix functional equations

I had the good fortune to work with Elliott Lieb at the Massachusetts Institute of Technology from 1968 to 1970, and looked at a number of Bethe ansatz problems. In particular I considered the most general inhomogeneous six-vertex model that could be solved by the Bethe ansatz, using equations of the same form, but involving different parameters at different sites of the lattice.[49] In particular, instead of the simple exponentials z_j^x above, one uses a more general “single-particle” function $z_j(x)$.

In 1970 my wife Elizabeth and I left MIT to return to Australia via England, where we would spent time with my parents in Essex. We intended to make the journey from England to Australia by ship. Already it was unusual to do this rather than travel by air (a change that had occurred only in the previous decade). We had the choice of two ships, giving us either two or five months' break in England.

We chose five months, and it was a good decision. Towards the end of that time I looked again at my MIT paper and realised that there were parameters in the final definition of the model that entered the eigenvalues of T , but *not* the eigenvectors. Indeed, this was apparent from Lieb and Sutherland's work on the zero-field six-vertex model: from (5.9) - (5.17), the eigenvectors f depend only on the single parameter $\Delta = -\cos \mu$. They are the same eigenvectors as those found by Yang and Yang[37] for the hamiltonian of the anisotropic Heisenberg chain.

The Boltzmann weights of the model depend also on v . If we write the transfer matrix T as $T(v)$ and keep Δ and μ fixed, then under quite general conditions it follows that two transfer matrices $T(u)$, $T(v)$, with the same value of Δ , *commute*:

$$T(u)T(v) = T(v)T(u) . \quad (5.21)$$

There is a common eigenvector matrix \mathcal{P} , independent of v , such that $\mathcal{P}^{-1}T(v)\mathcal{P}$ is diagonal, its entries being the eigenvalues $t(v)$.

Form a matrix $Q(v)$ such that $\mathcal{P}^{-1}Q(v)\mathcal{P}$ is also diagonal, with entries the corresponding $q(v)$ of (5.20). Then $T(v)$ and $Q(v)$ must also satisfy (5.20), i.e.

$$T(v)Q(v) = \phi(v)Q(v - \mu') + \phi(\mu - v)Q(v + \mu') , \quad (5.22)$$

the ϕ factors being scalars; $Q(v)$ must commute with $T(u)$, for all u, v .

This matrix relation, together with some elementary observations, is equivalent to the Bethe ansatz. Together with the commutation properties, it implies (5.20). Since the elements of T are sums of products of L Boltzmann weights, each equal to a, b or c , and vertices 5 and 6 always occur in pairs, each element must be of the form e^{-iLv} times a polynomial of degree L in e^{2iv} . If the elements of Q have a similar structure, but with L replaced by n , then the same must be true of the eigenvalues $t(v), q(v)$; and $q(v)$ must have n zeros β_1, \dots, β_n . Setting $v = \beta_j$ in (5.20) (and noting that $t(v)$ is entire, so finite), we obtain the Bethe equations (5.13).

6. Eight-vertex model

This was a new way of solving the six-vertex model. I knew from conversations with Elliott Lieb that an interesting generalization of the six-vertex model was the eight-vertex model, where one allows two more vertices, one with all arrows in, the other with all arrows out, with weights ω_7, ω_8 , respectively. §

§ Again, there is a special “free-fermion” case, when $\omega_1\omega_2 + \omega_3\omega_4 - \omega_5\omega_6 - \omega_7\omega_8 = 0$, which can be solved by pfaffians.[44]

This general eight-vertex model does not have the “conservation of down arrows” property that was necessary for the Bethe ansatz, but an obvious question was whether it has the commuting transfer matrix property. If so, then could one extend the above T, Q relation (5.22) to include this model, and so calculate κ and the free energy?

It turned out that the answer to both questions was yes, provided we restrict consideration to the zero-field model, with

$$\omega_1 = \omega_2 = a \quad , \quad \omega_3 = \omega_4 = b \quad , \quad \omega_5 = \omega_6 = c \quad , \quad \omega_7 = \omega_8 = d \quad . \quad (6.1)$$

(The last two equalities are not restrictions, as ω_7 and ω_8 must occur in pairs, as do ω_5 and ω_6 .) If we define

$$\Delta = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)} \quad , \quad \Gamma = \frac{ab - cd}{ab + cd} \quad , \quad (6.2)$$

then the transfer matrices of two models with different values of a, b, c, d , but the same values of Δ and Γ , commute.

Sutherland had shown in 1970[50] that each transfer matrix commutes with the hamiltonian of the XYZ chain (with coefficients Δ, Γ), which implies the required commutation relation between transfer matrices. I was unaware of this, but verified it myself and went on to construct the matrix $Q(u)$. This also satisfies (5.22), but now $\phi(u)$ is a product of Jacobi elliptic theta functions, as must be the eigenvalues $T(u), Q(u)$. They are doubly periodic in the complex u -plane and entire; $\phi(u)$ and $T(u)$ have L zeros per period parallelogram, while $Q(u)$ has $L/2$. Again, (5.22) is sufficient to determine the eigenvalues $T(u), Q(u)$. The results for the critical exponents were unusual and excited interest - they vary continuously with a parameter μ of the model that corresponds to the μ of the six-vertex model. For example, the critical exponent α of the specific heat is

$$\alpha = 2 - \frac{\pi}{\mu} \quad .$$

The eight-vertex model includes the Ising and six-vertex models as special cases. The Ising case is $\mu = \pi/2$, $\Delta = 0$, corresponding to the logarithmic singularity; $\mu = 0$, $\Delta = -1$ corresponds to the F model transition, with $\alpha = -\infty$; and $\mu = \pi$, $\Delta = 1$ to the KDP model transition, with $\alpha = 1$.

Of course, all this took many weeks to work out, and some of it was done on the ship from England to Australia. I shall always wonder if I would have had this idea if I had not had that five months’ break in England - it can be a good idea to give the mind time to relax and go on auto-pilot.

As with the Ising model, it was harder matter to obtain the spontaneous magnetization and spontaneous polarization. I obtained the polarization in 1973 for the six-vertex model in the anti-ferroelectric regime $\Delta < -1$ by a direct calculation of scalar products of Bethe eigenvectors: the method was somewhat tortuous, as is indicated by the fact that the proof of one step depended on taking n to be a

prime number! Michael Barber and I conjectured the spontaneous magnetization of the eight-vertex model in that year,[51] and Stewart Kelland and I the spontaneous polarization in 1974,[52] but derivations had to wait for the development of the corner transfer matrix method in 1976.[53]

7. Yang-Baxter relation

Two-dimensional nearest-neighbour lattice models can be grouped in three classes:

- Spins live on sites and interact along edges (e.g the Ising model) .
- Spins live on edges and interact at a vertex (e.g. the six and eight-vertex models).
- Spins live on sites and all four spins (on the square lattice) round a face interact (the eight-vertex model can also be formulated this way).

The condition for two transfer matrices to commute is a local one, involving the Boltzmann weights W of particular edges, vertices or faces. It takes different forms[17],[43, eqns. 9.68, 13.3.1] for the three above cases, and these can be represented graphically as in Figure 5. In each case W_1 can be interpreted as the weight (edge, vertex or face) of one transfer matrix, W_2 of the other, and W_3 as a supplementary weight in the equation. Each figure represents the partition function of a small graph. The outer spins (on the open circle sites or the six exterior edges of the second figure) are fixed and the inner ones (solid circles or the three edges of the inner triangles) are to be summed. The partition functions on each side must be equal, for all values of the corresponding exterior spins.

The first form is the star-triangle relation, which was used by Onsager.[1] The second was used by McGuire.[54]

7.1. Other models

This general technique turned out to be very useful in identifying and solving statistical mechanical models: some two-dimensional examples are

- Hard-hexagon model in 1980.[11]
- Fateev-Zamolodchikov model in 1982.[12]
- Kashiwara-Miwa model in 1986.[13]
- Solvable N -state chiral Potts model in 1988.[17]

In addition, the Yang-Baxter equations can be straightforwardly extended to three dimensions. The first solution to the resulting “tetrahedron” equations was given by Zamolodchikov in 1981.[14] Further solutions were found later, e.g by Bazhanov and Baxter.[15] As a general rule these three dimensional models are critical and their weights are not necessarily real and positive. This contrasts with

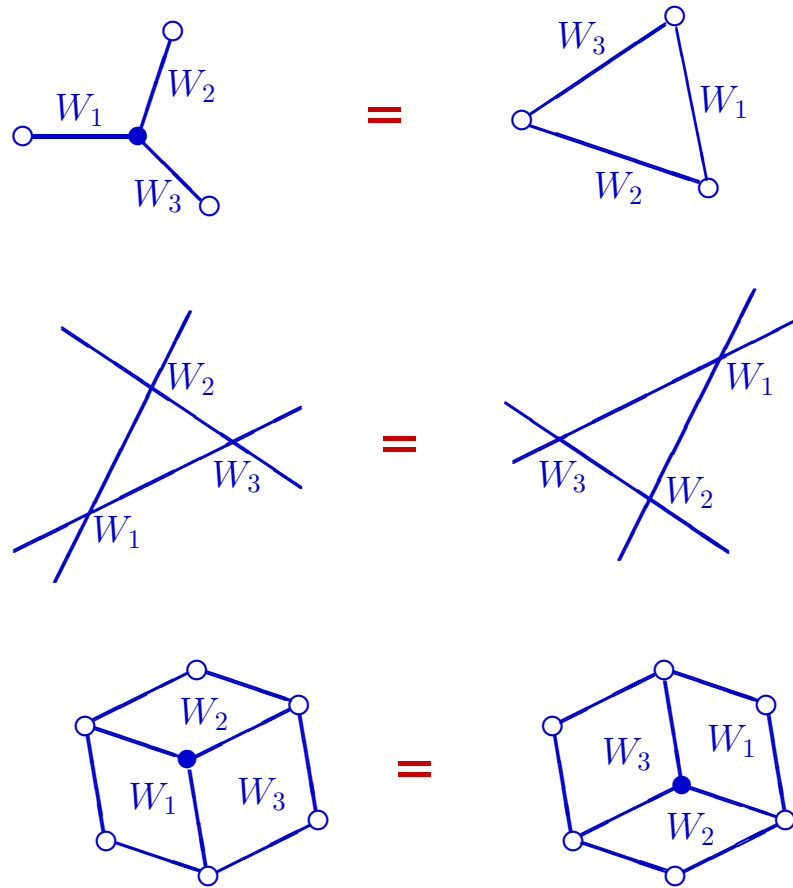


Figure 5. The three forms of the Yang-Baxter relation.

(say) the hard hexagon model, which is a good model of a triangular lattice gas, exhibiting the different phases, and with critical exponents that can (because of universality) be compared directly with experiment.[\[55\]](#)

7.2. Mathematical difficulties

Of all the models mentioned, the Ising model is undoubtedly the simplest. Because of its underlying free-fermion structure, all the eigenvalues can be evaluated explicitly, even for a finite lattice width L . This contrasts with the other models, where one usually does not know any explicit solutions of the non-linear T, Q (or equivalent) relation, and the best one can do is to use analytic techniques to calculate the largest and near-largest eigenvalues of T in the limit of large L . In general there

is no simple direct product property for the eigenvalues.

7.3. Rapidities

For all these two-dimensional solvable models, the Boltzmann weights W (and \overline{W}) depend on both the spin variables (e.g. a, b) and on other “rapidity” variables p, q :

$$W = W(a, b) = W_{pq}(a, b) \quad , \quad \overline{W} = \overline{W}(a, b) = \overline{W}_{pq}(a, b) \quad .$$

These rapidity variables are associated with the dotted lines in Figure 1. (For the moment ignore the break in the line below σ_1 .) Each edge of the original lattice of solid lines is intersected by two dotted lines, one horizontal and one vertical. The vertical (horizontal) dotted lines have rapidities p or p' (q or q'). In general each line may have its own rapidity variable, but for an homogeneous model all the vertical rapidities p must be the same, and similarly for the horizontal rapidities q .

The rapidity variables can be chosen so that two row-to-row transfer matrices, with different values q, q' of q , commute:

$$T_q T_{q'} = T_{q'} T_q \quad .$$

If the weight W_1 in Figure 5 depends on two rapidities q and r , and W_2 on r and p , then W_3 depends only on p and q : W_3 is independent of r .

The effect of the Yang-Baxter relation is that one can move these rapidity lines around without changing the partition function.[56] If these moves do not cross the spin σ_1 in Figure 1, then the RHS of (2.3) is also unchanged. If the lattice is infinitely big and σ_1 deep within it, then any particular rapidity line p (or q) can be moved infinitely far from σ_1 , where we do not expect p (or q) to contribute to the RHS of (4.1). It follows that the magnetization \mathcal{M} must be independent of *all* the rapidity variables, and indeed that is implied by the result (4.1). (For the Ising model, K and \overline{K} depend on the rapidities, but k is a constant, the same for all edges of the lattice.)

7.4. Rapidity difference property

With the exception of the solvable chiral Potts model, all the two-dimensional models mentioned above have the *rapidity difference property*, i.e. their weights W, \overline{W} depend on the corresponding rapidity variables p, q only via the difference $p - q$ (the variable u in (5.22) is such a difference). This is significant. It means that W, \overline{W} are trigonometric or elliptic functions of $p - q$. In the limit of a large system the partition function per site κ can be obtained easily by the “inversion relation” trick[57] and the spontaneous magnetization by the corner transfer matrix method.[43, chapter 13] These methods do involve assumptions about the analytic properties of the variables, for instance that κ is an analytic function of $u = p - q$ in some vertical strip in the complex u -plane.

8. Solvable Chiral Potts model

For the solvable chiral Potts model, $W(a, b) = W(a - b)$ and $\overline{W}(a, b) = \overline{W}(a - b)$, where

$$W(n) = \left(\frac{\mu_p}{\mu_q} \right)^n \prod_{j=1}^n \frac{y_q - \omega^j x_p}{y_p - \omega^j x_q}, \quad \overline{W}(n) = (\mu_p \mu_q)^n \prod_{j=1}^n \frac{\omega x_p - \omega^j x_q}{y_q - \omega^j y_p}. \quad (8.1)$$

Here

$$\omega = e^{2\pi i/N}, \quad (8.2)$$

μ_p, x_p, y_p are three complex variables related by the two equations

$$kx_p^N = 1 - k'/\mu_p^N, \quad ky_p^N = 1 - k'\mu_p^N, \quad (8.3)$$

and k, k' are constants satisfying

$$k^2 + k'^2 = 1. \quad (8.4)$$

We can think of $p = \{\mu_p, x_p, y_p\}$ as a point on a three-dimensional curve. This is the rapidity p . Similarly for q . If x_p, x_q, y_p, y_q lie on the unit circle and are ordered anti-clockwise in that sequence, then we can choose μ_p, μ_q so that the Boltzmann weights W, \overline{W} are real and positive. The conditions (8.3) for p, q ensure that

$$W(n + N) = W(n), \quad \overline{W}(n + N) = \overline{W}(n). \quad (8.5)$$

The model is therefore Z_N -symmetric. It is *chiral* because $W(-n) \neq W(n)$ and $\overline{W}(-n) \neq \overline{W}(n)$, so reflection-symmetry (left-hand equals right hand) is broken.

When $N = 2$ this is the Ising model, with $e^{-2K} = W(1)$, $e^{-2\overline{K}} = \overline{W}(1)$ (normalising the weights in (3.1) so that $W(0) = \overline{W}(0) = 1$).

For $N > 2$ this model does *not* have the ‘‘rapidity difference’’ property. This makes it much harder to calculate the thermodynamic properties. Even so, differential equations were written down [58] in 1988 that in principle define κ and the critical exponent α was thereby found to be $1 - 2/N$. An explicit expression as a double integral was obtained in 1990. [59, 60, 61]

It is natural to define the magnetization as

$$\mathcal{M}_r = \langle \omega^{r\sigma_1} \rangle = Z^{-1} \sum_{\sigma} \omega^{r\sigma_1} \prod_{\langle i, j \rangle} W(\sigma_i, \sigma_j) \quad (8.6)$$

for $r = 1, \dots, N - 1$.

In 1989 Albertini *et al* conjectured [62] that for $0 < k < 1$

$$\mathcal{M}_r = (1 - k^2)^{r(N-r)/2N^2}, \quad (8.7)$$

which is a beautifully simple formula that fits the $N = 2$ Ising case and the available series expansions. For $k > 1$ the system is disordered and the magnetization vanishes: $\mathcal{M}_r = 0$.

It was very difficult to prove this conjecture. In his collected works C.N. Yang says his 1952 calculation of the Ising model magnetization took six months and was

the longest in his career.[63] The chiral Potts magnetization took much longer: it was not till 2005, after spending much time on and off mulling over the problem, that I succeeded[64] in deriving the formula (8.7).

8.1. Broken rapidity line derivation of \mathcal{M}_r

The technique I used was invented by Jimbo, Miwa and Nakayashiki.[65] One breaks one of the dotted rapidity lines adjacent to the central spin σ_1 and gives the two halves different values q_1 and q_2 of the rapidity variable, as in Figure 1.

The effect of this is that one cannot remove these two half-lines away from σ_1 . One can still remove all the other rapidity lines, so now, using the definition (8.6),

$$\mathcal{M}_r = \text{function only of } k, q_1 \text{ and } q_2 . \quad (8.8)$$

This is a generalization of the magnetization.

However, one can rotate the two half-lines and cross them over. This gives functional relations for the generalized \mathcal{M}_r . For the models with the rapidity difference property, \mathcal{M}_r can only depend on q_1, q_2 via their difference. If one makes plausible assumptions about the analyticity properties of this function (e.g. analytic in a particular vertical strip), then one can solve the functional relations and obtain \mathcal{M}_r . This provides an alternative to the corner transfer matrix method of calculating single-spin correlations.

Again, however, life is more difficult for the chiral Potts model. It does not have the difference property and (ignoring the dependence on k , which we regard as a constant) \mathcal{M}_r is a function of two variables, not one. It is not obvious how to solve the functional relations, and what additional information is required.

What I observed in 2005 was that if one took

$$x_{q_2} = x_{q_1} , \quad y_{q_2} = \omega y_{q_1} , \quad \mu_{q_2} = \mu_{q_1} , \quad (8.9)$$

and again made a plausible analyticity assumption, then one could evaluate \mathcal{M}_r for this case.

Of course one wants it rather for the case when $q_2 = q_1$, but since it is then a constant, independent of q_1 , it is sufficient to obtain it at the intersection of these two cases, when

$$y_{q_2} = y_{q_1} = 0 . \quad (8.10)$$

In this way I was able to verify the conjecture (8.7). I still do not know what the generalized \mathcal{M}_r is for arbitrary q_1, q_2 .

9. Superintegrable Chiral Potts model

We return to the usual situation, where there are no broken rapidity lines and the magnetization \mathcal{M}_r of (8.6) is independent of the rapidities.

There is a special “superintegrable” case of the chiral Potts model, when the vertical rapidities alternate, as in Figure 1, taking the values p, p', p, p', \dots , where

$$x_{p'} = y_p, \quad y_{p'} = x_p, \quad \mu_{p'} = 1/\mu_p. \quad (9.1)$$

Since \mathcal{M} is independent of the rapidities, its value for the superintegrable case is the same as that of the general solvable chiral Potts model.

The chiral Potts model may be the most difficult of the two-dimensional solvable models, but its superintegrable case has some remarkable simplifications. In fact if we impose cylindrical boundary conditions as in Figure 1, with the top and bottom spins fixed to zero, then it has properties similar to those of the Ising model. (For $N = 2$, the superintegrable case, like the general solvable case, is the general zero-field Ising model.)

With these boundary conditions, the partition function Z is given by (2.5) and \mathcal{M}_r by

$$\mathcal{M}_r = \frac{u^\dagger T^n S_r T^{n'} u}{Z}, \quad (9.2)$$

where n is the number of rows below σ_1 and n' the number above, so $n + n' = M$. The vector u is defined by (2.6). Also, S_r is a diagonal matrix with entries

$$(S_r)_{s,s'} = \omega^{r\sigma_1} \prod_{i=1}^L \delta(\sigma_i, \sigma'_i) \quad (9.3)$$

again writing the spin set $\{\sigma_1, \dots, \sigma_L\}$ as s .

9.1. Partition function Z

Define a set of vectors u_0, \dots, u_{N-1} with elements

$$(u_a)_s = \prod_{i=1}^L \delta(\sigma_i, a). \quad (9.4)$$

Then $u = u_0$, where u is the vector above, defined by (2.6).

Let R be the spin-shift matrix with elements

$$R_{s,s'} = \prod_{i=1}^L \delta(\sigma_i, \sigma'_i + 1). \quad (9.5)$$

Then $R^N = 1$ and the eigenvalues of R are $1, \omega, \dots, \omega^{N-1}$. Let \mathcal{V}_Q be the vector space of all vectors v such that $Rv = \omega^Q v$. Then $u_a = R^a u$ and

$$v_Q = N^{-1/2} \sum_{a=0}^{N-1} \omega^{-Qa} u_a \in \mathcal{V}_Q \quad (9.6)$$

and

$$Z = Z_0 + Z_1 + \dots + Z_{N-1}, \quad (9.7)$$

where

$$Z_Q = Z_Q(M) = v_Q^\dagger T^M v_Q. \quad (9.8)$$

These matrices are of dimensions N^L and \mathcal{V}_Q is of dimension N^{L-1} . However, if we repeatedly multiply v_Q by T , we generate a smaller sub-space, [18, 66] of dimension 2^m , where

$$m = m_Q = \left\lfloor \frac{(N-1)L - Q}{N} \right\rfloor \quad (9.9)$$

and we write $[x]$ for the integer part of x . Further, we can choose the basis vectors (independently of q and k) so that

$$T = t_1 \otimes t_2 \otimes \cdots \otimes t_m \quad (9.10)$$

and

$$v_Q = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \quad (9.11)$$

Here each t_j is a two-by-two matrix. So to calculate Z_Q we only need T in a 2^m -dimensional sub-space, within which it is a direct product, as in the Ising model.

It follows that $Z_Q = Z_Q(M)$ is a product of m simple factors.

9.2. Magnetization

From (9.2) and (9.6),

$$\mathcal{M}_r = \frac{W_{0,r} + \cdots + W_{N-1,r}}{Z_0 + \cdots + Z_{N-1}} , \quad (9.12)$$

where

$$W_{Q,r} = v_Q^\dagger T^n S_r T^{n'} v_{Q+r} . \quad (9.13)$$

When $N = 2$ we regain the Ising model and we know then [4, 5, 19] for finite L that $W_{Q,r}$ can be written as a determinant. We have just seen that the superintegrable chiral Potts resembles the Ising model in that Z_Q is a simple product. We therefore ask whether our $W_{Q,r}$ can be written as a determinant for all N ?

The answer is yes. It can be written as a determinant. The working is given in a series of papers (not necessarily in logical order). [20, 21, 22, 23, 70]

Setting $m' = m_{Q+r}$, we find that

$$W_{Q,r} = Z_Q(n) Z_{Q+r}(n') \det[1 + AA^T] , \quad (9.14)$$

where A is an m by m' matrix.

This is huge progress. If $n = 3$ and $L = 15$, then $m = m' = 9$. We have reduced the problem from one calculating the elements of powers of the transfer matrix T , of dimension 14,349,907, to one of evaluating a nine by nine determinant!

Even so, we do want to take the large lattice limit. It is easy to allow n, n' to become infinite - we merely take the limit of the elements of the matrix A . We then want to let $L \rightarrow \infty$.

In this limit even m and m' will become infinite, so we need a way of evaluating the determinant in (9.14).

This was a tricky problem and I mulled over it for more than a year. If $m = m'$, then A is square and invertible, so

$$\det[1 + AA^T] = \det A \det[(A^T)^{-1} + A] . \quad (9.15)$$

The elements of A are of the form

$$A_{ij} = \frac{x_i x'_j}{c_i - c'_j} , \quad (9.16)$$

where x_i, x'_j, c_i, c'_j are known parameters.

The matrix A is therefore Cauchy-like and its determinant is [67]

$$\det A = \frac{XX' \prod_{1 \leq i < j \leq m} (c_i - c_j)(c_j - c'_i)}{\prod_{i=1}^m \prod_{j=1}^m (c_i - c'_j)} , \quad (9.17)$$

where $X = \prod_i x_i$, $X' = \prod_j x'_j$. Further, the inverse of the transpose of a Cauchy-like matrix is also Cauchy-like, the elements also being of the form (9.16), with only x_i, x'_j changed to some values y_i, y'_j .

The elements of the desired sum are therefore

$$\left((A^T)^{-1} + A\right)_{ij} = \frac{y_i y'_j + x_i x'_j}{c_i - c'_j} . \quad (9.18)$$

This is *not* a Cauchy-like matrix, rather it is akin to a Pick matrix of displacement rank 2. [68] There are fast computational algorithms for numerically calculating the determinants of such matrices, [69, Chapter 1] but I know of no explicit expression for the answer.

However, I finally looked at the numerator in the desired limit $n, n' \rightarrow \infty$. It is a rational function of λ_i, λ'_j , where

$$c_i = (1 + k'^2 - \lambda_i^2)/2k' , \quad c'_j = (1 + k'^2 - \lambda_j'^2)/2k' \quad (9.19)$$

so

$$c_i - c'_j = (\lambda_j'^2 - \lambda_i^2)/2k' . \quad (9.20)$$

Further, the numerator of (9.18) turns out to be of the form

$$y_i y'_j + x_i x'_j = s_i s'_j (\lambda_i + \lambda'_j) \quad (9.21)$$

so the factor $\lambda_i + \lambda'_j$ *cancels out* of (9.18), leaving

$$\left((A^T)^{-1} + A\right)_{ij} = \frac{-2k' s_i s'_j}{\lambda_i - \lambda'_j} . \quad (9.22)$$

This *is* a Cauchy-like matrix and one can obtain its determinant from the general formula (9.17). In this way one can obtain $W_{Q,r}/Z_Q(n)Z_{Q+r}(n')$ as a double product over $i, j = 1, \dots, m$, for *finite* L .

In general m, m' differ by at most one. If they are different, one can add a row or column to A so as to make it square and still Cauchy-like, while leaving $1 + AA^T$

unchanged. The same cancellation then occurs in (9.18) and again one obtains the double product expression for $W_{Q,r}$.

Finally one lets $m, m' \rightarrow \infty$ and evaluates the double product, using a Wiener-Hopf factorization. We again of course obtain the formula (8.7).

So this provides a rigorous proof of the magnetization of the superintegrable chiral Potts model, and hence of the Ising model. It does not involve Szegő's theorem. The full calculation is given in refs. [19] – [23] and [70]. A related derivation of the magnetization via the pair correlation function is given in [71].

10. Summary

There have been many developments in the statistical mechanics of lattice models since Onsager's famous solution of the Ising model in 1944. The general trend has been one of increasing complexity, first to models without the direct product transfer matrix property, then to the chiral Potts model where one loses also the useful rapidity-difference property.

But if one uses cylindrical boundary conditions with fixed spins on the top and bottom rows, then the wheel comes full circle with the superintegrable case of the chiral Potts model. One obtains determinantal expressions for the magnetization, analogous to those found for the Ising model by Onsager [24], Yang [4] and Montroll, Potts and Ward [5].

One can evaluate these determinants, not by calculating matrix eigenvalues, nor by using Szegő's theorem, but by identifying the determinants as Cauchy-like.

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